THE BETA-HERMITE AND BETA-LAGUERRE PROCESSES

Luen-Chau Li

Abstract. In this work, we introduce matrix-valued diffusion processes which describe the non-equilibrium situation of the matrix models for the beta-Hermite and the beta-Laguerre ensembles. We also study the corresponding spectral measure process and empirical eigenvalue/singular value process with regard to their limit laws.

1. Introduction.

The beta-ensembles have the physical interpretation as ensembles of one dimensional Coulomb gas in a neutralizing background with appropriate charge density [Dy,F], where β is the inverse temperature. For $\beta = 1, 2, 4$, matrix models of the beta-ensembles existed for a long time and are known as the $*$-orthogonal, $*$-unitary, and $*$-symplectic ensemble respectively, where $* \in \{$ Gaussian, Laguerre, Jacobi $\}$. (See for example, [Meh] and [Muir] and the references therein.) In recent years, as a result of the work of Dumitriu and Edelman [DE], and that of Killip and Nenciu [KN], matrix models for the beta-ensembles with entries/parameters from classical distributions are now known for all values of $\beta > 0$.

In this work, we will restrict ourselves to considerations which are related to the matrix models of the β-Hermite and the β-Laguerre (Wishart) ensembles. In order to explain what we want to do, let us consider the matrix model of the β-Hermite ensembles [DE], defined schematically by

$$J_\beta \sim \begin{pmatrix} \frac{1}{\sqrt{\beta}}N(0,1) & \frac{1}{\sqrt{2\beta}}\chi_{(n-1)\beta} & 0 & \cdots \\ \frac{1}{\sqrt{2\beta}}\chi_{(n-1)\beta} & \frac{1}{\sqrt{\beta}}N(0,1) & \frac{1}{\sqrt{2\beta}}\chi_{(n-2)\beta} & \ddots \\ 0 & \frac{1}{\sqrt{2\beta}}\chi_{(n-2)\beta} & \frac{1}{\sqrt{\beta}}N(0,1) & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{1.1}$$

where the entries on the diagonal and the subdiagonal of the $n \times n$ matrix J_β independent of one another. From the definition in (1.1), the joint density of the

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independent entries of J_β is given by

$$W_h(a,b) = c_{n\beta} \prod_{k=1}^{n} b_{n-k}^{k\beta-1} \exp\left[-\frac{\beta}{2}\left(\sum_{i=1}^{n} a_i^2 + 2\sum_{i=1}^{n-1} b_i^2\right)\right], \tag{1.2}$$

where $a_i = (J_\beta)_{ii}$, $b_i = (J_\beta)_{i,i+1}$, and $c_{n\beta}$ is a normalization constant whose value is given in (3.4). In this work, one of our motivating questions is the following: Is there a natural Markov process on Jacobi matrices for which $W_h(a,b)$ is the stationary steady state density ? Our approach to this question can be described as follows. Since the entries on the diagonal and the subdiagonal of J_β in (1.1) are independent random variables, therefore, it is natural to seek a Markov process which is a product of statistically independent Markov processes on the diagonal and the subdiagonal of the matrix. In this way, the problem reduces to that of constructing one-dimensional Markov processes with the required properties. As it turns out, for the class of probability density functions $p(x)$ satisfying Pearson's equation [P] (with support on an interval $I \subset \mathbb{R}$ with endpoints x_1 and x_2, say), the problem of constructing a class of stationary Markov processes with transition probability density $p(t, x_0, x)$ for which

$$\lim_{t\to\infty} p(t, x_0, x) = \int_{x_1}^{x_2} p(x_0) p(t, x_0, x) dx_0 = p(x), \tag{1.3}$$

has been the subject of investigation in [W]. The Pearson's family of probability density functions is quite broad, indeed it includes many of the continuous distributions which we commonly use such as normal, Chi-square, beta, etc. For us, although the Chi distribution is not on the list, however, the solution of the problem for Chi-square suffices. This is because the Markov process corresponding to the pdf of the Chi-square distribution can be identified to be the square of a generalized Bessel process. Hence the correponding process for the Chi distribution must be the generalized Bessel process itself and indeed, the transition probability density function for this process also satisfy the requirement in (1.3) above. (See Section 2.1 below.) As for the normal distribution with mean 0, it is quite well-known that the Ornstein-Uhlenbeck (OU) process describes the non-equilibrium situation [N]. In fact, one can readily understand Dyson's introduction of the matrix-valued Ornstein-Uhlenbeck processes in [Dy] from the perspective which we discussed above. Once the matrix-valued diffusion processes are in place, there are of course many questions one can ask about the statistics of the corresponding eigenvalue processes. In this work, we will begin with basic results on

the spectral measure and empirical eigenvalue processes. (In the case of the β-Laguerre process, we will instead deal with the empirical singular value process.) Thus in a sense, we are trying to understand the non-equilibrium properties of the Coulomb gas for general values of $\beta > 0$ and the these processes are basic. In this connection, we remark that for $\beta = 1, 2, 4$, the eigenvalue processes induced from the corresponding β-Hermite diffusion processes are actually different from those in [Dy], although there appears to be some subtle connections. We hope to return to this and other aspects of these processes in subsequent publications.

The paper is organized as follows. In Section 2, for the convenience of the reader, we begin with a review of the generalized Bessel and other related processes that play a key role in this work. In Section 3, after a short description on the product of independent Markov processes, we introduce the β-Hermite processes $(J_\beta(t))_{t\geq 0}$ for general values of $\beta > 0$ in the first subsection. Then we calculate the distribution of the eigenvalues and the first components of the normalized eigenvectors of $J_\beta(t)$ for each $t > 0$. The latter quantity, as it turns out, is independent of t. For the spectral measure $\mu_t = \sum_{j=1}^n \mu_j(t)\delta_{\lambda_j(t)}$ associated with $J_\beta(t)$, we also give the distribution of $\sum_{j=1}^k \mu_j(t)$ for each $1 \leq k \leq n$. In the second subsection, we consider the scaled process $(J_\beta^{(n)}(t))_{t\geq 0}$, where $J_\beta^{(n)}(t) = \frac{J_\beta(t)}{\sqrt{n}}$. The goal is to investigate the weak convergence (as $n \to \infty$), in probability, of the corresponding spectral measure μ_t^n and the empirical eigenvalue distribution ν_t^n for each $t > 0$. Because of the well-known one-to-one correspondence between Jacobi matrices and spectral measures (see, for example, [D1]), it is natural to consider μ_t^n. In this connection, we make use of the method of moments to show that μ_t^n converges weakly, in probability, to a deterministic measure μ_t whose density function has the shape of a semicircle. In this way, we obtain a time-dependent semicircle law. Because the distribution of the first components of the normalized eigenvectors of $J_\beta(t)$ is independent of t and the distribution of $\sum_{j=1}^k \mu_j(t)$ is in fact identical to the one in [BNR], we can make use of the proof of Theorem 5.4 in [BNR] to conclude that $d_{LP}(\mu_t^n, \nu_t^n)$ converges in probability to 0 as n tends to infinity. (Here d_{LP} is the Lévy-Prohorov metric.) As the authors are studying a rather different problem in [BNR], it is quite remarkable that we can make use of their analysis here. In Section 4, we do the same for the β-Laguerre processes and the associated β-Wishart processes. In the case of the scaled β-Wishart processes, the limiting law of both the spectral measure and the empirical eigenvalue distribution is that of a time-dependent Marchenko-Pastur law with a hard edge at 0 and a soft upper edge at $4\rho(t)$, where $\rho(t) = 1 - e^{-t}$. Finally,for the β-Laguerre processes, we show that the limit law for the empirical

singular value distribution is that of a time-dependent quarter circle law.

2. The generalized Bessel processes.

Let $(b(t))_{t\geq 0}$ be standard Brownian motion on $\mathbb{R}$, and $\delta > 0$. The Bessel process $(\tilde{R}^\delta(t))_{t\geq 0}$ of dimension δ starting from 0 [RY] is the square root of the process defined by the SDE

$$du(t) = 2\sqrt{|u(t)|}\, db(t) + \delta\, dt, \;\; u(0) = 0, \tag{2.1}$$

where the point 0 is an instantaneous reflecting barrier for $0 < \delta < 2$ and is polar otherwise. When δ is a positive integer, it is well-known that $\tilde{R}^\delta(t)$ is the modulus of a standard Brownian motion in $\mathbb{R}^\delta$. For our purpose, recall that the the transition probability density function of $\tilde{R}^\delta(t)$ is given by

$$\tilde{p}_t^\delta(x_0, x) = t^{-1}\left(\frac{x}{x_0}\right)^{\frac{\delta}{2}-1} x \exp\left(-\frac{x_0^2+x^2}{2t}\right) I_{\frac{\delta}{2}-1}\left(\frac{xx_0}{t}\right) \;\; \text{for}\, x_0 > 0, \tag{2.2}$$

and

$$\tilde{p}_t^\delta(0, x) = 2^{-\frac{\delta}{2}+1} t^{-\frac{\delta}{2}}\, \Gamma\left(\frac{\delta}{2}\right)^{-1} x^{\delta-1} \exp\left(-\frac{x^2}{2t}\right), \tag{2.3}$$

where $\Gamma(x)$ is the Gamma function, and I_ν is the modified Bessel function of order ν. Since we have $\tilde{R}^\delta(0) = 0$ with our definition, it follows that the pdf $\tilde{p}_t^\delta(x)$ of $\tilde{R}^\delta(t)$ coincides with $\tilde{p}_t^\delta(0, x)$. We will also deal with the Ornstein-Uhlenbeck processes in this work. By definition, the one dimensional Ornstein-Uhlenbeck process with parameters (a, σ) starting from 0 is the solution of the SDE

$$dv(t) = -av(t)\, dt + \sigma\, db(t), \;\; v(0) = 0, \tag{2.4}$$

where $a > 0$ and $\sigma \in \mathbb{R} \setminus \{0\}$. By the Dambis-Dubins-Schwarz theorem [RY], $v(t)$ is a time-changed Brownian motion:

$$v(t) = \sigma e^{-at} w\left(\frac{e^{2at}-1}{2a}\right), \tag{2.5}$$

where $w(t)$ is a standard Brownian motion in $\mathbb{R}$. Consequently, the random variables $v(t)$ are Gaussian with mean zero and variance

$$\rho(t) = \frac{\sigma^2}{2a}(1 - e^{-2at}), \tag{2.6}$$

and the transition probability density function is given by

$$\widehat{p}_t(x_0, x) = \frac{1}{\sqrt{2\pi\rho(t)}} \exp\left(-\frac{(x - x_0 e^{-at})^2}{2\rho(t)}\right). \tag{2.7}$$

Note that if $\mathbf{v}(t) = (v_1(t), \cdots, v_d(t))$, where $v_1(t), \cdots, v_d(t)$ are independent one dimensional Ornstein-Uhlenbeck processes starting from 0 with the same parameters (a, σ), then from (2.5) above, we have

$$v_i(t) = \sigma e^{-at} w_i\left(\frac{e^{2at} - 1}{2a}\right), \quad i = 1, \cdots, d, \tag{2.8}$$

where $w_1(t), \cdots, w_d(t)$ are independent standard Brownian motions on $\mathbb{R}$. Thus it follows from (2.8) that the radial part of the d-dimensional Ornstein-Uhlenbeck process $\mathbf{v}(t)$ is given by

$$\begin{aligned} R^d(t) :&= \sqrt{(v_1(t))^2 + \cdots + (v_d(t))^2} \\ &= \sigma e^{-at} \tilde{R}^d\left(\frac{e^{2at} - 1}{2a}\right), \end{aligned} \tag{2.9}$$

where $\tilde{R}^d$ is a Bessel process of dimension d starting from 0. Thus this motivates the following definition.

Definition 2.2. Let $\delta > 0$, and let a and σ be as above. If $(\tilde{R}^\delta(t))_{t\geq 0}$ is a Bessel process of dimension δ starting from 0, then the process $(R^\delta(t))_{t\geq 0}$ defined by

$$R^\delta(t) = \sigma e^{-at} \tilde{R}^\delta\left(\frac{e^{2at} - 1}{2a}\right) \tag{2.10}$$

is called the generalized Bessel process of dimension δ starting from 0 with parameters (a, σ).

From the above definition, a straightforward calculation using (2.2) and (2.3) shows that the pdf and the transition probability density function of $R^\delta(t)$ are given by

$$p_t^\delta(x) = p_t^\delta(0, x) = 2^{-\frac{\delta}{2}+1} \rho(t)^{-\frac{\delta}{2}}\, \Gamma\left(\frac{\delta}{2}\right)^{-1} x^{\delta-1} \exp\left(-\frac{x^2}{2\rho(t)}\right) 1_{[0,\infty)}(x) \tag{2.11}$$

and

$$\begin{aligned} &p_t^\delta(x_0, x) \\ =&\rho(t)^{-1} \left(\frac{x}{e^{-at} x_0}\right)^{\frac{\delta}{2}-1} x \exp\left[-\frac{(x^2 + e^{-2at} x_0^2)}{2\rho(t)}\right] I_{\frac{\delta}{2}-1}\left(\frac{x e^{-at} x_0}{\rho(t)}\right),\ x_0 > 0. \end{aligned} \tag{2.12}$$

Remark 2.3. In [Eie], the author used the term generalized Bessel process to denote the radial part of a d-dimensional Ornstein-Uhlenbeck process. Here we are adapting this terminology to the more general situation when δ is not necessarily an integer.

If $\widehat{p}_t(x)$ denote the pdf of the Ornstein-Uhlenbeck process above, it is well-known that [N]

$$\lim_{t\to\infty} \widehat{p}_t(x_0, x) = \int_{-\infty}^{\infty} \widehat{p}_\infty(x_0)\widehat{p}_t x_0, x)dx_0 = \widehat{p}_\infty(x), \tag{2.13}$$

where $\widehat{p}_\infty(x)$ is the pdf of the normal distribution $N(0, \rho(\infty))$.

Proposition 2.4. *For the transition probability density function of the generalized Bessel process in (2.11),(2.12),*

$$\lim_{t\to\infty} p_t^\delta(x_0, x) = \int_0^\infty p_\infty^\delta(x_0)p_t^\delta(x_0, x)dx_0 = p_\infty^\delta(x), \tag{2.14}$$

where

$$p_\infty^\delta(x) = 2^{-\frac{\delta}{2}+1}\rho(\infty)^{-\frac{\delta}{2}+1}\Gamma\left(\frac{\delta}{2}\right)^{-1} x^{\delta-1}\exp\left(-\frac{x^2}{2\rho(\infty)}\right). \tag{2.15}$$

Proof. The transition probability density function $q_t^\delta(x_0, x)$ of the square of the generalized Bessel process above is related to $p_t^\delta(x_0, x)$ by

$$q_t^\delta(x_0, x) = \frac{1}{2\sqrt{x}} p_t^\delta(\sqrt{x_0}, \sqrt{x}). \tag{2.16}$$

Hence the explicit expression for $q_t^\delta(x_0, x)$ can be computed from (2.11) and (2.12). Note that the function $q_\infty^\delta(x) = 2^{-1}x^{-1/2}p_\infty^\delta(\sqrt{x})$ (resp. $q_t^\delta(x_0, x)$) is related to the one in Eqn.(28) of [W] (resp. Eqn.(30) of [W]) by scaling of the time variable, the spatial variables and an overall scaling of the function itself. Hence it follows from [W] that

$$\lim_{t\to\infty} q_t^\delta(x_0, x) = \int_0^\infty q_\infty^\delta(x_0)q_t^\delta x_0, x)dx_0 = q_\infty^\delta(x), \tag{2.17}$$

where

$$q_\infty^\delta(x) = 2^{-\frac{\delta}{2}}\rho(\infty)^{-\frac{\delta}{2}}\Gamma\left(\frac{\delta}{2}\right)^{-1} x^{\frac{\delta}{2}-1}\exp\left(-\frac{x}{2\rho(\infty)}\right). \tag{2.18}$$

Since $p_\infty^\delta(x) = 2x\, q_\infty^\delta(x^2)$, it is immediate from (2.17) that $\lim_{t\to\infty} p_t^\delta(x_0, x) = p_\infty^\delta(x)$. Moreover, on using (2.15) and (2.17), we obtain

$$\begin{aligned} \int_0^\infty p_\infty^\delta(x_0)p_t^\delta(x_0, x)\, dx_0 &= 2x\int_0^\infty q_\infty^\delta(y_0)q_t^\delta(y_0, x^2)\, dy_0 \\ &= 2xq_\infty^\delta(x^2) = p_\infty^\delta(x). \end{aligned} \tag{2.19}$$

So this completes the proof. □

Remark 2.5. (a) Alternatively, one could give a direct proof of the limiting behaviour of $p_t^\delta(x_0,x)$ for $x_0>0$ by invoking the asymptotics $I_\nu(z)\sim(z/2)^\nu/\Gamma(\nu+1)$ as $z\to 0$. On the other hand, the fact that $p_\infty^\delta(x)$ is the stationary density of the generalized Bessel process can also be established in a direct way by making use of the Weber Sonine formula [Wat]

$$\begin{aligned}&\int_0^\infty x^\mu e^{\alpha x^2} J_\nu(\gamma x)\,dx\\ &=\frac{\beta^\nu\Gamma\left(\frac{1}{2}(\mu+\nu+1)\right)}{2^{\nu+1}\alpha^{\frac{1}{2}(\mu+\nu+1)}\Gamma(\nu+1)}\,{}_1F_1\left(\frac{1}{2}(\mu+\nu+1),\nu+1;-\frac{\gamma^2}{4\alpha}\right),\end{aligned}\tag{2.20}$$

valid for $\operatorname{Re}\alpha>0$, $\operatorname{Re}(\mu+\nu)>-1$. We will leave the details to the interested reader.

(b) In [W], the author applies the method of separation of variables to the Fokker-Planck equation and identifies the cofficients of this equation with those of the Pearson's equation. In this way, eigenvalue problems of Sturm-Liouville type with reflecting boundary conditions at the end points are obtained. It is interesting to point out that as a result of the connection which we mentioned above, we have the expansion

$$\begin{aligned}&p_t^\delta(x_0,x)\\ &=p_\infty^\delta(x)\sum_{n=0}^\infty nB\left(n,\frac{\delta}{2}\right)e^{-2ant}L_n^{\frac{\delta}{2}-1}\left(\frac{x_0^2}{2\rho(t)}\right)L_n^{\frac{\delta}{2}-1}\left(\frac{x^2}{2\rho(t)}\right),\end{aligned}\tag{2.21}$$

where $B(a,b)$ is the Beta function and

$$L_n^\alpha(x)=\frac{1}{n!}x^{-\alpha}e^{-x}\frac{d^n}{dx^n}(x^\alpha e^{-x})\tag{2.22}$$

are the Laguerre polynomials.

For our construction in the rest of the paper, we will pick the normalization $(a,\sigma)=(1/2,1)$. In this case, $p_\infty^\delta(x)$ is just the pdf of the Chi distribution χ_δ while $\widehat{p}_\infty(x)$ is that of standard normal. Hence the corresponding generalized Bessel process and Ornstein-Uhlenbeck process have the desired properties. (See (1.1) above.)

3. The beta-Hermite processes.

We begin by defining the product Markov process which will be used in our construction in the present section and the next. For this purpose, let $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ be two statistically independent time-homogeneous Markov processes with transition probability density functions given by $p_X(t, x_0, x)$ and $p_Y(t, y_0, y)$ respectively. Then the product process $X \otimes Y$ is defined to be the process where $(X\otimes Y)_t = (X_t, Y_t)$ for $t \geq 0$. Put $p_{X\otimes Y}(t, (x_0, y_0), (x, y)) = p_X(t, x_0, x)p_Y(t, y_0, y)$, then it can be verified that the product process is again a Markov process with $p_{X\otimes Y}(t, (x_0, y_0), (x, y))$ as its transition probability density function. Clearly, we can iterate this construction and so we can define the product Markov process for any given number of statistically independent time-homogeneous Markov processes.

3.1 The beta-Hermite processes and their eigenvalue distribution.

Definition 3.1.1. The β-Hermite process $(J_\beta(t))_{t\geq 0}$ is the stochastic process on $n \times n$ Jacobi matrices $J_\beta(t)$ given by

$$J_\beta(t) = \begin{pmatrix} \frac{1}{\sqrt{\beta}}U_1(t) & \frac{1}{\sqrt{2\beta}}R^{(n-1)\beta}(t) & 0 & \cdots \\ \frac{1}{\sqrt{2\beta}}R^{(n-1)\beta}(t) & \frac{1}{\sqrt{\beta}}U_2(t) & \frac{1}{\sqrt{2\beta}}R^{(n-2)\beta}(t) & \ddots \\ 0 & \frac{1}{\sqrt{2\beta}}R^{(n-2)\beta}(t) & \frac{1}{\sqrt{\beta}}U_3(t) & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \tag{3.1}$$

where the processes on the diagonal and the subdiagonal are statistically independent of each other. Here, $U_1(t), \cdots, U_n(t)$ are Ornstein-Uhlenbeck proceeses starting from 0 with parameters $(1/2, 1)$ and $R^{j\beta}(t)$ is the generalized Bessel process of dimension $j\beta$ starting from 0 with parameters $(1/2, 1)$, $j = 1, \cdots, n-1$.

Proposition 3.1.2. *The β-Hermite process is a matrix-valued diffusion process starting from* 0 *with transition probability density function*

$$P(t, \widetilde{J}, J) = 2^{\frac{n}{2}}\beta^{n-\frac{1}{2}} \prod_{i=1}^{n} \widehat{p}_t(\sqrt{\beta}\ \tilde{a}_i, \sqrt{\beta}\ a_i) \prod_{j=1}^{n-1} p_t^{(n-j)\beta}(\sqrt{2\beta}\ \tilde{b}_j, \sqrt{2\beta}\ b_j), \tag{3.2}$$

with respect to Lebesgue measure $dadb = da_1 \cdots da_n db_1 \cdots db_{n-1}$ on $\mathbb{R}^n \times \mathbb{R}_+^{n-1}$, where $\tilde{a}_i = \widetilde{J}_{ii}$, $\tilde{b}_i = \widetilde{J}_{i,i+1}$, $a_i = J_{ii}$ and $b_i = J_{i,i+1}$. Consequently, the joint density of the independent entries of $J_\beta(t)$ is given by

$$P(t, 0, J) = c_{n\beta}\, \rho(t)^{-\frac{n}{2}-\frac{\beta}{4}n(n-1)} \prod_{k=1}^{n-1} b_{n-k}^{k\beta-1} \exp\left(-\frac{\beta}{2\rho(t)} trJ^2\right), \tag{3.3}$$

where

$$c_{n\beta} = \frac{2^{\frac{n}{2}-1}\beta^{\frac{n}{2}+\frac{\beta}{4}n(n-1)}}{\pi^{\frac{n}{2}}\prod_{k=1}^{n-1}\Gamma\left(\frac{k\beta}{2}\right)}. \tag{3.4}$$

Finally, if $W_h(a,b)$ is the joint density of the independent entries of the β-Hermite ensemble in (1.1), then

$$\lim_{t\to\infty} P(t,\widetilde{J},J) = \int_{\mathbb{R}^n\times\mathbb{R}^n_+} W_h(\tilde{a},\tilde{b})P(t,\widetilde{J},J)\ d\tilde{a}d\tilde{b} = W_h(a,b). \tag{3.5}$$

Proof. It is easy to see that the transition probability density functions of $\frac{1}{\sqrt{2\beta}}R^\delta(t)$ and $\frac{1}{\sqrt{\beta}}v(t)$ are given by $\sqrt{2\beta}\,p_t^\delta(\sqrt{2\beta}\,x_0,\sqrt{2\beta}\,x)$ and $\sqrt{\beta}\,\widehat{p}_t(\sqrt{\beta}\,x_0,\sqrt{\beta}\,x)$ respectively. Therefore, (3.2) is a consequence of the product construction of independent Markov processes. As $P[J_\beta(0)=0]=1$, the pdf in (3.3) now follows from (3.2), (2.11), and the explicit formula for $\widehat{p}_t(x)$. Finally, the validity of (3.5) is due to (2.13) and (2.14) and this completes the verification.

□

Our next goal is to calculate the joint probability density function of the eigenvalues of $J_\beta(t)$. In preparation, observe that the pdf of $\frac{1}{\sqrt{\beta}}U_i(t)$ can be expressed in terms of the pdf $\widehat{p}_\infty(x)$ of $N(0,1)$. Indeed, this pdf is given by

$$\begin{aligned}\sqrt{\beta}\,\widehat{p}_t(\sqrt{\beta}\,x) &= \left(\frac{\beta}{2\pi\rho(t)}\right)^{1/2}\exp\left(-\frac{\beta x^2}{2\rho(t)}\right)\\ &=\sqrt{\frac{\beta}{\rho(t)}}\ \widehat{p}_\infty\left(\sqrt{\frac{\beta}{\rho(t)}}\ x\right).\end{aligned} \tag{3.6}$$

Similarly, it follows from (2.11) that the pdf of $\frac{1}{\sqrt{2\beta}}R^{j\beta}(t)$ takes the form

$$\begin{aligned}\sqrt{2\beta}\,p_t^{j\beta}(\sqrt{2\beta}\,x) &=\frac{2\left(\frac{\beta}{\rho(t)}\right)^{j\beta/2}}{\Gamma\left(\frac{j\beta}{2}\right)}\ x^{j\beta-1}\exp\left(-\frac{\beta x^2}{\rho(t)}\right)\ 1_{[0,\infty)}(x)\\ &=\sqrt{\frac{2\beta}{\rho(t)}}\ p_\infty^{j\beta}\left(\sqrt{\frac{2\beta}{\rho(t)}}\ x\right),\end{aligned} \tag{3.7}$$

where $p_\infty^{j\beta}(x)$ is the pdf of the Chi distribution $\chi_{j\beta}$. Hence we have

$$\begin{aligned}P(t,0,J) &= 2^{\frac{n}{2}}\left(\frac{\beta}{\rho(t)}\right)^{n-\frac{1}{2}}\prod_{i=1}^n\widehat{p}_\infty(\sqrt{\beta/\rho(t)}\ a_i)\prod_{j=1}^{n-1}p_\infty^{(n-j)\beta}(\sqrt{2\beta/\rho(t)}\ b_j)\\ &=\frac{1}{\rho(t)^{n-\frac{1}{2}}}W_h\left(\frac{a}{\sqrt{\rho(t)}},\frac{b}{\sqrt{\rho(t)}}\right).\end{aligned} \tag{3.8}$$

Now recall that a generic Jacobi matrix $J = J(a, b)$ has nonzero entries on the subdiagonal and the eigenvalues of J are distinct. We will order the eigenvalues such that $\lambda_1 > \lambda_2 > \cdots \lambda_n$ and denote by $f_1(1) > 0, f_2(1) > 0, \cdots, f_n(1) > 0$ the first components of the normalized eigenvectors corresponding to the distinct eigenvalues. Put $\lambda = (\lambda_1, \cdots, \lambda_n)$, $f(1) = (f_1(1), \cdots, f_n(1))$. It is well-known that the map $\phi : J(a, b) \longrightarrow (\lambda, f(1))$ is a diffeomorphism from the set of generic Jacobi matrices with positive subdiagonal entries to $C_+ \times S_+^{n-1}$, where $C_+ = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_1 > \cdots > x_n\}$, and S_+^{n-1} consists of vectors $q = (q_1, \cdots, q_n)$ on the unit sphere S^{n-1} such that $q_i > 0$ for all i (see, for example, [D1]). Moreover, it follows from [DE],[D2] that

$$da\, db = \frac{\prod_{i=1}^{n-1} b_i}{\prod_{i=1}^{n} f_i(1)} d\lambda\, d\sigma, \tag{3.9}$$

where

$$d\sigma = \frac{df_1(1) \cdots df_{n-1}(1)}{f_n(1)} \tag{3.10}$$

is the element of surface area in S_+^{n-1}.

Theorem 3.1.3. *Under the β-Hermite process, the vector $\lambda(t) = (\lambda_1(t), \cdots, \lambda_n(t))$ of ordered eigenvalues of $J_\beta(t)$ and the vector $f(1, t) = (f_1(1, t), \cdots, f_n(1, t))$ of first components of normalized eigenvectors are independent. If $\Delta(\lambda)$ is the Vandermonde determinant, then*

(a) the joint pdf of the unordered eigenvalues of $J_\beta(t)$ is given by

$$\begin{aligned} P_{n\beta}(t, \lambda) &= C_{n\beta}\, \rho(t)^{-\frac{n}{2} - \frac{\beta}{4} n(n-1)} |\Delta(\lambda)|^\beta \exp\left(-\frac{\beta}{2\rho(t)} \sum_{i=1}^{n} \lambda_i^2\right) \\ &= C_{n\beta}\, \rho(t)^{-\frac{n}{2} - \frac{\beta}{4} n(n-1)} \exp(-\beta W(t, \lambda)), \end{aligned} \tag{3.11}$$

where

$$C_{n\beta} = (2\pi)^{-\frac{n}{2}} \beta^{\frac{n}{2} + \frac{\beta}{4} n(n-1)} \prod_{j=1}^{n} \frac{\Gamma\left(1 + \frac{\beta}{2}\right)}{\Gamma\left(1 + \frac{j\beta}{2}\right)}, \tag{3.12}$$

and

$$W(t, \lambda) = \frac{1}{2\rho(t)} \sum_{i=1}^{n} \lambda_i^2 - \sum_{i<j} \log|\lambda_i - \lambda_j|. \tag{3.13}$$

(b) the joint density of $f_1(1, t), \cdots, f_n(1, t)$ with respect to the measure $d\sigma$ on S_+^{n-1} is given by

$$2^{n-1} \frac{\Gamma\left(\frac{n\beta}{2}\right)}{\left(\Gamma\left(\frac{\beta}{2}\right)\right)^n} \prod_{i=1}^{n} q_i^{\beta-1}. \tag{3.14}$$

Proof. Let ϕ be the map above and consider $J_\beta(t)$ in the domain of ϕ. Let $\lambda_1(t) > \cdots > \lambda_n(t)$ be the (ordered) eigenvalues of $J_\beta(t)$ and let $f_1(1,t) > 0, \cdots f_n(1,t) > 0$ be the first components of the normalized eigenvectors corresponding to the eigenvalues. Because of (3.8), the independence of $\lambda(t)$ and $f(1,t)$ follows as in the calculation in [DE] where we have to use (3.9) and the relation [D1],[DE]

$$\Delta(\lambda) \equiv \prod_{i<j}(\lambda_i - \lambda_j) = \frac{\prod_{i=1}^{n-1} b_{n-i}^i}{\prod_{i=1}^n f_i(1)}. \tag{3.15}$$

The calculations leading to the assertions in the remaining parts of the proposition are also similar and so we skip the details. □

Now, another way to describe generic Jacobi matrices $J(a,b)$ is by using the spectral measure

$$\mu = \sum_{j=1}^n \mu_j\, \delta_{\lambda_j}, \quad \mu_j = f_j(1)^2,\ \ 1 \le j \le n. \tag{3.16}$$

Indeed, it is well-known that the map $\psi : J(a,b) \longrightarrow \mu$ is a bijection from the set of generic $n \times n$ Jacobi matrices to the set of probability measures on $\mathbb{R}$ supported at n points. Let

$$\mu(t) = (\mu_1(t), \cdots, \mu_n(t)), \quad \mu_j(t) = f_j(1,t)^2,\ 1 \le j \le n. \tag{3.17}$$

To describe the probability distribution of the vector $\mu(t)$, recall that the Dirichlet distribution $\mathrm{Dir}_{n-1}(\alpha_1, \cdots; \alpha_n)$ with parameters $\alpha_1, \cdots, \alpha_n > 0$ is the distribution which has a density with respect to Lebesgue measure on $\mathbb{R}^{n-1}$ given by (see [Wi] for more details)

$$\frac{\Gamma(\alpha_1 + \cdots + \alpha_n)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)} \prod_{j=1}^n y_j^{\alpha_j - 1}\, 1_S(y_1, \cdots, y_{n-1}) \tag{3.18}$$

where

$$y_n = 1 - y_1 - \cdots - y_{n-1}, \tag{3.19}$$

and where S is the simplex

$$S = \left\{ (y_1, \cdots, y_{n-1}) \middle|\ y_j \ge 0 \text{ for all } j, \sum_{j=1}^{n-1} y_j \le 1 \right\}. \tag{3.20}$$

Note that for $n = 2$, $\mathrm{Dir}_1(\alpha_1; \alpha_2)$ is the Beta distribution $\mathrm{Beta}(\alpha_1, \alpha_2)$ which is supported on $[0,1]$. The following result is a straightforward consequence of the joint density of $f_1(1,t), \cdots, f_n(1,t)$ with respect to $d\sigma$ and the basic properties of the Dirichlet distribution [Wi].

Corollary 3.1.4. *Under the β-Hermite process, the vector of weights $\mu(t)$ of the spectral measure associated with $J_\beta(t)$ follows the distribution $Dir_{n-1}(\frac{\beta}{2}, \cdots ; \frac{\beta}{2})$. Hence the marginals are Beta distributions:*

$$\mu_j(t) \sim Beta\left(\frac{\beta}{2}, \frac{(n-1)\beta}{2}\right), \ 1 \leq j \leq n. \tag{3.21}$$

Moreover, for each $1 \leq k \leq n$,

$$\sum_{j=1}^{k} \mu_j(t) \sim Beta\left(\frac{k\beta}{2}, \frac{(n-k)\beta}{2}\right). \tag{3.22}$$

3.2 Time-dependent semicircle law.

We introduce the following scaling of $J_\beta(t)$:

$$J_\beta^{(n)}(t) = \frac{J_\beta(t)}{\sqrt{n}}, \tag{3.23}$$

and let $a_k^{(n)}(t) = (J_\beta^{(n)}(t))_{kk}$, $b_k^{(n)}(t) = (J_\beta^{(n)}(t))_{k,k+1}$. The goal of this subsection is to study the large n behaviour of the spectral measure process

$$\mu_t^n = \sum_{j=1}^{n} \mu_j^{(n)}(t)\delta_{\lambda_j^{(n)}(t)} \tag{3.24}$$

and the empirical eigenvalue process

$$\nu_t^n = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i^{(n)}(t)} \tag{3.25}$$

for each $t > 0$, where $\lambda_1^{(n)}(t), \cdots, \lambda_n^{(n)}(t)$ are the eigenvalues of $J_\beta^{(n)}(t)$. We begin with a lemma.

Lemma 3.2.1. *As $n \to \infty$,*

$$a_k^{(n)}(t) \xrightarrow{\mathrm{P}} 0, \quad b_k^{(n)}(t) \xrightarrow{\mathrm{P}} \sqrt{\frac{\rho(t)}{2}} \tag{3.26}$$

for each $t > 0$.

Proof. For any $t > 0$ and any $r = 1, 2, \cdots$, it follows from (3.6) that

$$\begin{aligned} E[(a_k^{(n)}(t))^r] &= \int_{-\infty}^{\infty} \left(\frac{n\beta}{2\pi\rho(t)}\right)^{1/2} x^r \exp\left(-\frac{n\beta x^2}{2\rho(t)}\right) dx \\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{\rho(t)}{n\beta}\right)^{\frac{r}{2}} \int_{-\infty}^{\infty} x^r e^{-\frac{x^2}{2}}\, dx \end{aligned} \tag{3.27}$$

from which it is clear that $E[(a_k^{(n)}(t))^r] \to 0$ as $n \to \infty$. Therefore, $a_k^{(n)}(t) \xrightarrow{\mathrm{d}} 0$ and hence $a_k^{(n)}(t) \xrightarrow{\mathrm{P}} 0$. Similarly, we obtain from (3.7) that

$$\begin{aligned} E[(b_k^{(n)}(t))^r] &= \frac{2\left(\frac{n\beta}{\rho(t)}\right)^{(n-k)\beta/2}}{\Gamma\left(\frac{(n-k)\beta}{2}\right)} \int_0^\infty x^{(n-k)\beta-1+r} \exp\left(-\frac{n\beta x^2}{\rho(t)}\right) dx \\ &= \frac{2\left(\frac{\rho(t)}{n\beta}\right)^{\frac{r}{2}}}{\Gamma\left(\frac{(n-k)\beta}{2}\right)} \int_0^\infty x^{(n-k)\beta+r-1} e^{-x^2}\, dx \qquad (3.28) \\ &= \left(\frac{\rho(t)}{n\beta}\right)^{\frac{r}{2}} \frac{\Gamma\left(\frac{(n-k)\beta+r}{2}\right)}{\Gamma\left(\frac{(n-k)\beta}{2}\right)}. \end{aligned}$$

From the asymptotics of the Gamma function, we have

$$\frac{\Gamma\left(\frac{(n-k)\beta+r}{2}\right)}{\Gamma\left(\frac{(n-k)\beta}{2}\right)} \sim \left(\frac{n\beta}{2}\right)^{\frac{r}{2}} \tag{3.29}$$

as $n \to \infty$. Hence the assertion $b_k^n(t) \xrightarrow{\mathrm{P}} \sqrt{\frac{\rho(t)}{2}}$ follows from (3.28) and (3.29). $\square$

We now turn to the analysis of the spectral measure process $(\mu_t^n)_{t \geq 0}$. First, from the relations

$$\int_{\mathbb{R}} x^k d\mu_t^n(x) = (e_1, (J_\beta^{(n)}(t))^k\, e_1),\ \ k = 1, \cdots, \tag{3.30}$$

it is clear that the moments $\int_{\mathbb{R}} x^k d\mu_t^n(x)$ are polynomials in the entries of $J_\beta^{(n)}(t)$. As the entries on the diagonal and subdiagonal of $J_\beta^{(n)}(t)$ are independent random variables, it follows from (3.26) that as $n \to \infty$,

$$(a_1^{(n)}(t), b_1^{(n)}(t), \cdots, a_j^{(n)}(t), b_j^{(n)}(t)) \xrightarrow{\mathrm{d}} \left(0, \sqrt{\frac{\rho(t)}{2}}, \cdots, 0, \sqrt{\frac{\rho(t)}{2}}\right) \tag{3.31}$$

for each fixed value of j and each $t > 0$. Hence by the continuous mapping theorem, (3.30) and (3.31), we obtain

$$\int_{\mathbb{R}} x^k d\mu_t^n(x) \xrightarrow{\mathrm{P}} (e_1, (J^{(\infty))}(t))^k\, e_1) \tag{3.32}$$

as $n \to \infty$, where $J^{(\infty)}(t)$ is the Jacobi operator on ℓ_2^+ given by

$$J^{(\infty))}(t) = \begin{pmatrix} 0 & \sqrt{\frac{\rho(t)}{2}} & 0 & 0 & \cdots \\ \sqrt{\frac{\rho(t)}{2}} & 0 & \sqrt{\frac{\rho(t)}{2}} & 0 & \cdots \\ 0 & \sqrt{\frac{\rho(t)}{2}} & 0 & \sqrt{\frac{\rho(t)}{2}} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}. \tag{3.33}$$

Now the orthogonal polynomials corresponding to $J^{(\infty)}(t)$ are defined by

$$P_n^t(x) = \frac{\sin(n\theta)}{\sin\theta}, \quad x = \sqrt{2\rho(t)}\cos\theta \tag{3.34}$$

and it is easy to check that

$$\int_{\mathbb{R}} P_m^t(x) P_n^t(x)\, d\mu_t(x) = \delta_{mn}, \tag{3.35}$$

where

$$d\,\mu_t(x) = \frac{\sqrt{2\rho(t) - x^2}}{\pi\rho(t)}\, 1_{[\,-\sqrt{2\rho(t)},\sqrt{2\rho(t)}\,]}(x)\, dx. \tag{3.36}$$

Thus $d\mu_t$ is the spectral measure of $J^{(\infty)}(t)$ so that

$$(e_1, (J^{(\infty)}(t))^k\, e_1) = \int_{\mathbb{R}} x^k d\mu_t(x), \tag{3.37}$$

Combining (3.32) and (3.37), we obtain the first part of the following theorem.

Theorem 3.2.2. *(a) For each $t > 0$, the sequence $(\mu_t^n)_{n\geq 1}$ converges weakly, in probability, to the probability measure μ_t defined in (3.36).*
(b) For each $t > 0$, the sequence $(\nu_t^n)_{n\geq 1}$ converges weakly, in probability, to the same probability measure μ_t.

Proof. We have already proved (a). In order to establish (b), it suffices to show that

$$d_{LP}(\mu_t^n, \nu_t^n) \xrightarrow{\mathrm{P}} 0 \quad \text{as } n \to \infty, \tag{3.38}$$

where d_{LP} is the Lévy-Prohorov metric on the space of (Borel) probability measures on $\mathbb{R}$. For this purpose, introduce the distribution functions $F_{\mu_t^n}$, $F_{\nu_t^n}$ corresponding to μ_t^n and ν_t^n respectively. Then from the definitions of the Lévy-Prohorov metric and the Lévy distance between distribution functions, we have

$$\begin{aligned} d_{LP}(\mu_t^n, \nu_t^n) &\leq \sup |F_{\mu_t^n}(x) - F_{\nu_t^n}(x)| \\ &\leq \max \left| \sum_{j=1}^k \mu_j^{(n)}(t) - \frac{k}{n} \right|, \end{aligned} \tag{3.39}$$

and so it suffices to show that

$$\max \left| \sum_{j=1}^k \mu_j^{(n)}(t) - \frac{k}{n} \right| \xrightarrow{\mathrm{P}} 0 \quad \text{as } n \to \infty. \tag{3.40}$$

As $\sum_{j=1}^{k} \mu_j^{(n)}(t)$ has the Beta distribution (see (3.22)), the rest of the proof of identical to that of Theorem 5.4 in [BNR]. For the sake of completeness, we give the main steps. First of all, from the density of the Beta distribution, we have the moments of $\sum_{j=1}^{k} \mu_j^{(n)}(t)$:

$$E\left[\left(\sum_{j=1}^{k} \mu_j^{(n)}(t)\right)^r\right] = \frac{\Gamma\left(\frac{k\beta}{2}+r\right)\Gamma\left(\frac{n\beta}{2}\right)}{\Gamma\left(\frac{k\beta}{2}\right)\Gamma\left(\frac{n\beta}{2}+r\right)} \tag{3.41}$$

from which we see that

$$E\left[\sum_{j=1}^{k} \mu_j^{(n)}(t)\right] = \frac{k}{n}. \tag{3.42}$$

By using (3.41) and its special case in (3.42), we find

$$E\left[\left|\sum_{j=1}^{k} \mu_j^{(n)}(t) - \frac{k}{n}\right|^4\right] = O\left(\frac{k(n-k)}{n^4}\right) \tag{3.43}$$

so that

$$\sum_{k=1}^{n} E\left[\left|\sum_{j=1}^{k} \mu_j^{(n)}(t) - \frac{k}{n}\right|^4\right] = O\left(\frac{1}{n}\right). \tag{3.44}$$

Hence for each $\epsilon > 0$, we obtain

$$\begin{aligned} P\left[\max\left|\sum_{j=1}^{k} \mu_j^{(n)}(t) - \frac{k}{n}\right| > \epsilon\right] &\le \sum_{k=1}^{n} P\left[\left|\sum_{j=1}^{k} \mu_j^{(n)}(t) - \frac{k}{n}\right| > \epsilon\right] \\ &\le \epsilon^{-4}\sum_{k=1}^{n} E\left[\left|\sum_{j=1}^{k} \mu_j^{(n)}(t) - \frac{k}{n}\right|^4\right] \\ &= O\left(\frac{1}{n\epsilon^4}\right) \end{aligned} \tag{3.45}$$

from which (3.40) follows.

□

4. The beta-Laguerre and the beta-Wishart processes.

The matrix model of the β-Laguerre ensembles parametrized by $a > -1$ [DE] is defined schematically by

$$L_{\beta,a} \sim \begin{pmatrix} \frac{1}{\sqrt{\beta}}\chi_{(a+n)\beta} & \frac{1}{\sqrt{\beta}}\chi_{\beta(n-1)} & 0 & \cdots \\ 0 & \frac{1}{\sqrt{\beta}}\chi_{(a+n-1)\beta} & \frac{1}{\sqrt{\beta}}\chi_{(n-2)\beta} & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{4.1}$$

where the entries on the diagonal and the subdiagonal of the $n \times n$ matrix $L_{\beta,a}$ are independent. Using the pdf of the Chi distribution, the joint density of the independent entries of $L_{\beta,a}$ reads

$$W_l(x,y) = d_{n\beta} \prod_{i=1}^{n} x_i^{(a+n-i+1)\beta-1} e^{-\frac{\beta}{2}x_i^2} \prod_{i=1}^{n-1} y_i^{(n-i)\beta-1} e^{-\frac{\beta}{2}y_i^2}, \tag{4.2}$$

where $x_i = (L_{\beta,a})_{ii}$, $y_i = (L_{\beta,a})_{i,i+1}$ and

$$d_{n\beta} = \frac{2^{2n-1}(\beta/2)^{\frac{na\beta}{2}+\frac{\beta}{2}n^2}}{\prod_{j=1}^{n-1}\Gamma\left(\frac{j\beta}{2}\right)\prod_{j=1}^{n}\Gamma\left(\frac{(a+j)\beta}{2}\right)}. \tag{4.3}$$

In this section, we will present results analogous to those in Section 3 for two related matrix models: the one defined in (4.1) and an associated one consisting of matrices $L_{\beta,a}^T L_{\beta,a}$.

4.1 The beta Laguerre (Wishart) processes and the eigenvalue distribution.

Definition 4.1.1. The β-Laguerre process $(L_{\beta,a}(t))_{t\geq 0}$ parametrized by $a > -1$ is the stochastic process on $n \times n$ bidiagonal matrices $L_{\beta,a}(t)$ given by

$$L_{\beta,a}(t) = \begin{pmatrix} \frac{1}{\sqrt{\beta}}R^{(a+n)\beta}(t) & \frac{1}{\sqrt{\beta}}R^{(n-1)\beta}(t) & 0 & \cdots \\ 0 & \frac{1}{\sqrt{\beta}}R^{(a+n-1)\beta}(t) & \frac{1}{\sqrt{\beta}}R^{(n-2)\beta}(t) & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \tag{4.4}$$

where the processes on the diagonal and subdiagonal are statistically independent of each other, and where $R^\delta(t)$ is the generalized Bessel process of dimension δ starting from 0 with parameters $(1/2, 1)$. Let $J_{\beta,a}(t) = L_{\beta,a}(t)^T L_{\beta,a}(t)$, then the process $(J_{\beta,a}(t))_{t\geq 0}$ is called the associated β-Wishart process.

As in the analogous case in Proposition 3.1.2, the following is immediate from the product construction of independent Markov processes, $P[L_{\beta,a}(0) = 0] = 1$, (2.11) and (2.14).

Proposition 4.1.2. *The β-Laguerre process is the matrix-valued diffusion process starting from* 0 *with transition probability density function*

$$P(t,\widetilde{L},L)=\beta^{n-\frac{1}{2}}\prod_{i=1}^{n}p_t^{(a+n-i+1)\beta}(\sqrt{\beta}\ \tilde{x}_i,\sqrt{\beta}\ x_i)\prod_{j=1}^{n-1}p_t^{(n-i)\beta}(\sqrt{\beta}\ \tilde{y}_i,\sqrt{\beta}\ y_i),\tag{4.5}$$

with respect to Lebesgue measure $dxdy=dx_1\cdots dx_ndy_1\cdots dy_{n-1}$ on $\mathbb{R}_+^n\times\mathbb{R}_+^{n-1}$, where $\tilde{x}_i=\widetilde{L}_{ii}$, $\tilde{y}_i=\widetilde{L}_{i,i+1}$, $x_i=L_{ii}$ and $b_i=L_{i,i+1}$. Thus the joint density of the independent entries of $L_{\beta,a}(t)$ is given explicitly by

$$P(t,0,L)=d_{n\beta}\,\rho(t)^{-\frac{na\beta}{2}-\frac{\beta n^2}{2}}\prod_{i=1}^{n}x_i^{(a+n-i+1)\beta-1}e^{-\frac{\beta x_i^2}{2\rho(t)}}\prod_{i=1}^{n-1}y_i^{(n-i)\beta-1}e^{-\frac{\beta y_i^2}{2\rho(t)}}.\tag{4.6}$$

If $W_l(x,y)$ is the density in (4.1), we have

$$\lim_{t\to\infty}P(t,\widetilde{L},L)=\int_{\mathbb{R}_+^n\times\mathbb{R}_+^n}W_l(\tilde{x},\tilde{y})P(t,\widetilde{L},L)\ d\tilde{x}d\tilde{y}=W_l(x,y).\tag{4.7}$$

We next calculate the joint probability density function of the eigenvalues of $J_{\beta,a}(t)=L_{\beta,a}(t)^TL_{\beta,a}(t)$. To do that, we have to first compute the pushforward of the measure $P(t,0,L)dxdy$ under the map $L\mapsto J=L^TL$ which sends $n\times n$ bidiagonal matrices to $n\times n$ Jacobi matrices. To this end, observe that

$$\begin{aligned}P(t,0,L)&=\left(\frac{\beta}{\rho(t)}\right)^{n-\frac{1}{2}}\prod_{i=1}^{n}p_\infty^{(a+n-i+1)\beta}\left(\frac{x_i}{\sqrt{\rho(t)}}\right)\prod_{i=1}^{n-1}p_\infty^{(n-i)\beta}\left(\frac{y_i}{\sqrt{\rho(t)}}\right)\\&=\frac{1}{\rho(t)^{n-\frac{1}{2}}}W_l\left(\frac{x}{\sqrt{\rho(t)}},\frac{y}{\sqrt{\rho(t)}}\right).\end{aligned}\tag{4.8}$$

Therefore, if we let $a_i=J_{ii}=(L^TL)_{ii}$, $b_i=J_{i,i+1}=(L^TL)_{i,i+1}$, then in the notations of section 3 for Jacobi matrices, the calculations leading to the results in the next proposition are similar to the corresponding one in [DE] where one has to use (3.9), (3.15) and the fact that the Jacobian of the map $L\mapsto J=L^TL$ is given by [DE]

$$2^nx_n\prod_{i=1}^{n-1}x_i^2.\tag{4.9}$$

Theorem 4.1.3. *Under the β-Wishart process, the vector $\lambda(t)=(\lambda_1(t),\cdots,\lambda_n(t))$ of ordered eigenvalues of $J_{\beta,a}(t)$ and the vector $f(1,t)=(f_1(1,t),\cdots,f_n(1,t))$ of first components of normalized eigenvectors are independent. If $\Delta(\lambda)$ is the Vandermonde determinant, then*

(a) the joint pdf of the unordered eigenvalues of $J_{\beta,a}(t)$ is given by

$$P^a_{n\beta}(t,\lambda) = C^a_{n\beta}\,\rho(t)^{-\frac{\beta a n}{4}-\frac{\beta}{4}n^2}|\Delta(\lambda)|^\beta \prod_{i=1}^n \lambda_i^{\frac{\beta}{2}(a+1)-1}\exp\left(-\frac{\beta}{2\rho(t)}\sum_{i=1}^n \lambda_i^2\right), \quad (4.10)$$

where

$$C^a_{n\beta} = \left(\frac{\beta}{2}\right)^{\frac{\beta}{4}an+\frac{\beta}{4}n^2}\prod_{j=1}^n \frac{\Gamma\left(1+\frac{\beta}{2}\right)}{\Gamma\left(1+\frac{j\beta}{2}\right)\Gamma\left(\frac{(a+j)\beta}{2}\right)}. \quad (4.11)$$

(b) the joint density of $f_1(1,t),\cdots,f_n(1,t)$ with respect to the measure $d\sigma$ on S^{n-1}_+ is given by

$$2^{n-1}\frac{\Gamma\left(\frac{n\beta}{2}\right)}{\left(\Gamma\left(\frac{\beta}{2}\right)\right)^n}\prod_{i=1}^n q_i^{\beta-1}. \quad (4.12)$$

In view of (4.12), it follows that if $\mu(t) = (\mu_1(t),\cdots,\mu_n(t))$ is the vector of weights in the spectral measure of the spectral measure of $J_{\beta,a}(t)$, then the distributions of $\mu_j(t)$ and $\sum_{j=1}^k \mu_j(t)$ are also given by (3.21) and (3.22) respectively.

4.2 Time-dependent Marchenko-Pastur law and quarter-circle law.

We introduce the following scaling of $L_{\beta,a}(t)$:

$$L^{(n)}_{\beta,a}(t) = \frac{L_{\beta,a}(t)}{\sqrt{n}}, \quad (4.13)$$

and let $x^{(n)}_k(t) = (L^{(n)}_{\beta,a}(t))_{kk}$, $y^{(n)}_k(t) = (L^{(n)}_{\beta,a}(t))_{k,k+1}$.

We begin with the analog of Lemma 3.2.1 for the β-Laguerre process.

Lemma 4.2.1. *As $n\to\infty$,*

$$x^{(n)}_k(t) \xrightarrow{\mathrm{P}} \sqrt{\rho(t)}, \quad y^{(n)}_k(t) \xrightarrow{\mathrm{P}} \sqrt{\rho(t)}, \quad (4.14)$$

for each $t>0$.

Proof. For any $t > 0$ and any $r = 1, 2, \cdots$, it follows from the pdf of $R^\delta(t)$ that

$$
\begin{aligned}
E[(y_k^{(n)}(t))^r] =& \frac{2\left(\frac{n\beta}{2\rho(t)}\right)^{\frac{(n-k)\beta}{2}}}{\Gamma\left(\frac{(n-k)\beta}{2}\right)} \int_0^\infty x^{(n-k)\beta+r} \exp\left(\frac{-\beta n x^2}{2\rho(t)}\right) dx \\
=& \frac{2\left(\frac{2\rho(t)}{n\beta}\right)^{\frac{r}{2}}}{\Gamma\left(\frac{(n-k)\beta}{2}\right)} \int_0^\infty x^{(n-k)\beta-1+r} e^{-x^2} dx \\
=& \frac{\left(\frac{2\rho(t)}{n\beta}\right)^{\frac{r}{2}}}{\Gamma\left(\frac{(n-k)\beta}{2}\right)} \cdot \Gamma\left(\frac{(n-k)\beta+r}{2}\right) \\
\sim& \rho(t)^{r/2}
\end{aligned}
\tag{4.15}
$$

as $n \to \infty$ where we have used the asymptotics of the Gamma function. Hence $y_k^{(n)}(t) \xrightarrow{\text{P}} \sqrt{\rho(t)}$ as $n \to \infty$. Similarly,

$$
\begin{aligned}
E[(x_k^{(n)}(t))^r] &= \frac{2\left(\frac{n\beta}{2\rho(t)}\right)^{\frac{(a+n-k+1)\beta}{2}}}{\Gamma\left(\frac{(a+n-k+1)\beta}{2}\right)} \int_0^\infty x^{(a+n-k+1)\beta+r-1} \exp\left(\frac{-\beta n x^2}{2\rho(t)}\right) dx \\
&= \left(\frac{2\rho(t)}{n\beta}\right)^{\frac{r}{2}} \cdot \frac{\Gamma\left(\frac{(a+n-k+1)\beta+r}{2}\right)}{\Gamma\left(\frac{(a+n-k+1)\beta}{2}\right)} \\
&\sim \rho(t)^{r/2}
\end{aligned}
\tag{4.16}
$$

as $n \to \infty$ and so we also have $x_k^{(n)}(t) \xrightarrow{\text{P}} \sqrt{\rho(t)}$. □

Now we introduce

$$J_{\beta,a}^{(n)}(t) = (L_{\beta,a}^{(n)}(t))^T L_{\beta,a}^{(n)}(t) \tag{4.17}$$

and let $a_k^{(n)}(t) = (J_{\beta,a}^{(n)}(t))_{kk}$, $b_k^{(n)}(t) = (J_{\beta,a}^{(n)}(t))_{k,k+1}$. Then from the relations between the entries of $J_{\beta,a}^{(n)}(t)$ and $L_{\beta,a}^{(n)}(t)$, we can deduce the following from Proposition 4.2.1 when we invoke the continuous mapping theorem.

Corollary 4.2.2. *As $n \to \infty$,*

$$
\begin{aligned}
&a_1^{(n)}(t) \xrightarrow{\text{P}} \rho(t), \quad a_k^{(n)}(t) \xrightarrow{\text{P}} 2\rho(t),\ k > 1, \\
&b_k^{(n)}(t) \xrightarrow{\text{P}} \rho(t),\ k \geq 1
\end{aligned}
\tag{4.18}
$$

for each $t > 0$.

Let $\lambda_1^{(n)}(t), \cdots, \lambda_n^{(n)}(t)$ be the eigenvalues of $J_{\beta,a}^{(n)}(t)$. We consider the spectral measure process

$$\mu_t^n = \sum_{j=1}^{n} \mu_j^{(n)}(t)\delta_{\lambda_j^{(n)}(t)} \tag{4.19}$$

and the empirical eigenvalue process

$$\nu_t^n = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i^{(n)}(t)} \tag{4.20}$$

associated with $J_{\beta,a}^{(n)}(t)$. From the relations

$$\int_{\mathbb{R}} x^k d\mu_t^n(x) = (e_1, (J_{\beta,a}^{(n)}(t))^k\, e_1),\ \ k = 1, \cdots, \tag{4.21}$$

it follows that the moments $\int_{\mathbb{R}} x^k d\mu_t^n(x)$ are polynomials in the entries of $J_{\beta,a}^{(n)}(t)$. Thus it follows from Corollary 4.2.2 and the continuous mapping theorem that

$$\int_{\mathbb{R}} x^k d\mu_t^n(x) \xrightarrow{\mathrm{P}} (e_1, (J_{\beta,a}^{(\infty))}(t))^k\, e_1) \tag{4.22}$$

as $n \to \infty$, where $J_{\beta,a}^{(\infty)}(t)$ is the Jacobi operator on ℓ_2^+ given by

$$J_{\beta,a}^{(\infty))}(t) = \begin{pmatrix} \rho(t) & \rho(t) & 0 & 0 & \cdots & & \\ \rho(t) & 2\rho(t) & \rho(t) & 0\cdots & & & \\ 0 & \rho(t) & 2\rho(t) & \rho(t) & 0 & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & & \end{pmatrix}. \tag{4.23}$$

The next thing to do is to compute the spectral measure $d\mu_t$ of $J_{\beta,a}^{(\infty)}(t)$. For this purpose, we will make use of the method of Grosjean [G]. First of all, we introduce the (time-dependent) polynomials $\{P_n^t(x)\}_{n\geq 0}$ satisfying the recursion relations

$$P_{n+1}^t(x) = (x - 2\rho(t))P_n^t(x) - \rho(t)^2 P_{n-1}^t(x),\ \ n \geq 1 \tag{4.24}$$

and the initial conditions

$$P_0^t(x) = 1, \quad P_1^t(x) = x - \rho(t). \tag{4.25}$$

Also, introduce the functions of the second kind

$$Q_n^t(z) = \int_{\mathbb{R}} \frac{P_n^t(x)}{z - x}\, d\mu_t(x) \tag{4.26}$$

for $n \geq 0$ and for $z \in \mathbb{C}$. From (4.24) and (4.25) above, we have

$$Q_1^t(t) = -1 + (z - \rho(t))Q_0^t(z), \tag{4.27}$$

and

$$Q_{n+1}^t(z) = (z - 2\rho(t))Q_n^t(z) - \rho(t)^2 Q_{n-1}^t(z), \ n \geq 1, \tag{4.28}$$

where $Q_0^t(z)$ is the Stieltjes transform of the spectral measure $d\mu_t$. By making use of these relations, we obtain the continued fraction expansion

$$Q_0^t(z) = \frac{1}{z - \rho(t) - F^t(z)}, \tag{4.29}$$

where

$$F^t(z) = \cfrac{\rho(t)^2}{z - 2\rho(t) - \cfrac{\rho(t)^2}{z - 2\rho(t) - \cfrac{\rho(t)^2}{z - 2\rho(t) - \dots}}}. \tag{4.30}$$

But from the expression for $F^t(z)$, it is clear that

$$F^t(z) = \frac{\rho(t)^2}{z - 2\rho(t) - F^t(z)}. \tag{4.31}$$

Solving, we obtain

$$F^t(z) = \frac{z - 2\rho(t) - \sqrt{(z - 2\rho(t))^2 - 4\rho(t)^2}}{2}, \tag{4.32}$$

where the branch of the square root is the one which tends to the positive square root of $(x - 2\rho(t))^2 - 4\rho(t)^2$ when z tends to $x \in (4\rho(t), \infty)$. Therefore, when we substitute (4.32) into (4.29), the result is

$$\int_{\mathbb{R}} \frac{d\mu_t(x)}{z - x} = \frac{1}{\frac{z}{2} + \frac{1}{2}\sqrt{(z - 2\rho(t))^2 - 4\rho(t)^2}}. \tag{4.33}$$

We next introduce the Fourier transform of $d\mu_t$:

$$\widehat{\mu}_t(x) = \int_{\mathbb{R}} e^{ixs} \, d\mu_t(s). \tag{4.34}$$

Then from (4.33), its Laplace transform is given by

$$\begin{aligned} \mathcal{L}(\widehat{\mu}_t)(p) &= \int_0^\infty e^{-xp} \widehat{\mu}_t(x) \, dx \\ &= -i \int_{\mathbb{R}} \frac{d\mu_t(s)}{-ip - s} \\ &= \frac{i}{\frac{ip}{2} - \frac{1}{2}\sqrt{(ip + 2\rho(t))^2 - 4\rho(t)^2}} \end{aligned} \tag{4.35}$$

for Re $(p) > 0$. Thus by the inversion theorem for Laplace transform, we have

$$\frac{1}{2\pi}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{e^{xp}}{\frac{ip}{2}-\frac{1}{2}\sqrt{(ip+2\rho(t))^2-4\rho(t)^2}}\,dp=\begin{cases}0, & x>0\\ \frac{1}{2}\widehat{\mu}_t(0+), & x=0\\ \widehat{\mu}_t(x), & x>0,\end{cases} \tag{4.36}$$

where σ is a real number which is greater than the real parts of the singularities of $\mathcal{L}(\widehat{\mu}_t)(p)$ in the p-plane. Now make the change of variable $p = iu$ in the integral on the left hand side of (4.36), this gives

$$\frac{-i}{2\pi}\int_{-\infty-i\sigma}^{\infty-i\sigma}\frac{e^{ixu}}{\frac{u}{2}+\frac{1}{2}\sqrt{(u-2\rho(t))^2-4\rho(t)^2}}=\begin{cases}0, & x>0\\ \frac{1}{2}\widehat{\mu}_t(0+), & x=0\\ \widehat{\mu}_t(x), & x>0,\end{cases} \tag{4.37}$$

where the path of integration is now a horizontal line below the singularities of the integrand in the u-plane. But from the above expression, it is easy to show that the denominator of the integrand vanishes only at $u = 0$. Thus the set of singularities of the integrand coincides with the branch cut $[0, 4\rho(t)]$ of the function $\sqrt{(u-2\rho(t))^2-4\rho(t)^2}$ and hence we can take $\sigma = -\epsilon$, where $\epsilon > 0$ is a small number. But the fact that $d\mu_t$ is a real measure means that $\overline{\widehat{\mu}_t(-x)} = \widehat{\mu}_t(x)$ for all x. Consequently,

$$\widehat{\mu}_t(x)=\frac{i}{2\pi}\int_{\mathbb{R}}\left[\frac{e^{ixu}}{\frac{u}{2}+\frac{1}{2}\sqrt{u(u-4\rho(t))}}\right]_{u=s-i\epsilon}^{u=s+i\epsilon}ds, \tag{4.38}$$

for all $x \in \mathbb{R}$ where $\widehat{\mu}_t(0) = \frac{1}{2}(\widehat{\mu}_t(0+) + \widehat{\mu}_t(0-))$. But from the definition of $\sqrt{u(u-4\rho(t))}$, we have

$$\lim_{\epsilon\to 0+}\sqrt{u(u-4\rho(t))}\,|_{u=s\pm i\epsilon}=\begin{cases}-\sqrt{s(s-4\rho(t))}, & s\in(-\infty,0]\\ \pm i\sqrt{s(4\rho(t)-s)}, & s\in(0,4\rho(t))\\ \sqrt{s(s-4\rho(t))}, & s\in[4\rho(t),\infty).\end{cases} \tag{4.39}$$

Therefore, when we take the limit as $\epsilon \to 0+$ in (4.38), the only contribution to the integral comes from $[0, 4\rho(t)]$ and we find

$$\widehat{\mu}_t(x)=\int_0^{4\rho(t)}e^{ixs}\frac{1}{2\pi\rho(t)}\sqrt{\frac{4\rho(t)-s}{s}}\,ds. \tag{4.40}$$

Hence we can now conclude that

$$d\mu_t(x)=\frac{1}{2\pi\rho(t)}\sqrt{\frac{4\rho(t)-x}{x}}\,1_{(0,4\rho(t))}\,dx \tag{4.41}$$

and so we have the following result.

Theorem 4.2.3. *(a) For each $t > 0$, the sequence $(\mu_t^n)_{n\geq 1}$ converges weakly, in probability, to the probability measure μ_t defined in (4.41).*
(b) For each $t > 0$, the sequence $(\nu_t^n)_{n\geq 1}$ converges weakly, in probability, to the same probability measure μ_t.

We now return to the β-Laguerre process itself. Note that although we have Theorem 4.2.3 available to us, however, it is not hard to see that it is not possible to deduce the corresponding result for the β-Laguerre process. Our study of the β-Laguerre process will be based on the following transformation which is well-known in numerical linear algebra [GK]. Suppose B is the bidiagonal matrix

$$B = \begin{pmatrix} x_1 & y_1 & & & \bigcirc \\ & x_2 & y_2 & & \\ & & \ddots & \ddots & \\ & & & x_{n-1} & y_{n-1} \\ \bigcirc & & & & x_n \end{pmatrix} \tag{4.42}$$

with singular value decomposition $B = U\Sigma V^T$, where $U = (u_1, \cdots, u_n)$ and $V = (v_1, \cdots, v_n)$ are orthogonal, and $\Sigma = \operatorname{diag}(\sigma_1, \cdots, \sigma_n)$ is the diagonal matrix whose diagonal entries are the singular values of B. Then the eigenvalues of the $2n \times 2n$ symmetric tridiagonal matrix

$$S = \begin{pmatrix} 0 & x_1 & & & & \bigcirc \\ x_1 & 0 & y_1 & & & \\ & y_1 & 0 & & & \\ & & & & \ddots & \\ & & & \ddots & & \\ & & & & & x_n \\ \bigcirc & & & & x_n & 0 \end{pmatrix} \tag{4.43}$$

are $\sigma_1, \cdots, \sigma_n, -\sigma_1, \cdots, -\sigma_n$ and the first components of the corresponding normalized eigenvectors are given by $v_1(1)/\sqrt{2}, \cdots, v_n(1)/\sqrt{2}$, $v_1(1)/\sqrt{2}$, $\cdots$, $v_n(1)/\sqrt{2}$ respectively. In view of this, it suffices to study the process $(S_{\beta,a}(t))_{t\geq 0}$, where

$$S_{\beta,a}(t) = \begin{pmatrix} 0 & \frac{1}{\sqrt{\beta}}R^{(a+n)\beta}(t) & 0 & \cdots \\ \frac{1}{\sqrt{\beta}}R^{(a+n)\beta}(t) & 0 & \frac{1}{\sqrt{\beta}}R^{(n-1)\beta}(t) & \ddots \\ 0 & \frac{1}{\sqrt{\beta}}R^{(n-1)\beta}(t) & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \tag{4.44}$$

is obtained from (4.4) by applying the above transformation. (The full justification is in Theorem 4.2.4 below.) We introduce the following scaling of $S_{\beta,a}(t)$:

$$S_{\beta,a}^{(n)}(t) = \frac{S_{\beta,a}(t)}{\sqrt{n}}, \tag{4.45}$$

corresponding to (4.13). By our discussion above, if $\sigma_1^{(n)}(t), \cdots, \sigma_n^{(n)}(t)$ denote the singular values of $L_{\beta,a}(t)$, then the eigenvalues of $S_{\beta,a}^{(n)}(t)$ are given by $\sigma_1^{(n)}(t), \cdots,$ $\sigma_n^{(n)}(t)$, $-\sigma_1^{(n)}(t), \cdots, -\sigma_n^{(n)}(t)$. Now we introduce the spectral measure process

$$\widetilde{\mu}_t^{2n} = \sum_{j=1}^{n} \widetilde{\mu}_j^n(t) \left(\delta_{\sigma_j^{(n)}(t)} + \delta_{-\sigma_j^{(n)}(t)} \right) \tag{4.46}$$

and the empirical eigenvalue process

$$\widetilde{\nu}_t^{2n} = \frac{1}{2n} \sum_{i=1}^{n} \left(\delta_{\sigma_j^{(n)}(t)} + \delta_{-\sigma_j^{(n)}(t)} \right) \tag{4.47}$$

associated with $(S_{\beta,a}^{(n)}(t))_{t\geq 0}$, where

$$\widetilde{\mu}_j^n(t) = \frac{1}{2} f_j(1,t)^2, \quad j = 1, \cdots, n. \tag{4.48}$$

By using Lemma 4.2.1, and following the same procedure in Section 3.2, we obtain

$$\int_{\mathbb{R}} x^k d\, \widetilde{\mu}_t^{2n}(x) \xrightarrow{\mathrm{P}} (e_1, (\sqrt{2} J^{(\infty)})(t))^k\, e_1) = \int_{\mathbb{R}} x^k d\widetilde{\mu}_t(x), \tag{4.49}$$

where $J^{(\infty)}$ is the Jacobi operator in (3.33) and

$$d\, \widetilde{\mu}_t(x) = \frac{\sqrt{4\rho(t) - x^2}}{2\pi\rho(t)} 1_{[\,-2\sqrt{\rho(t)}, 2\sqrt{\rho(t)}\,]}(x)\, dx. \tag{4.50}$$

Thus we have proved

$$d_{LP}(\widetilde{\mu}_t^{2n}, \widetilde{\mu}_t) \xrightarrow{\mathrm{P}} 0 \quad \text{as } n \to \infty. \tag{4.51}$$

We next show that

$$d_{LP}(\widetilde{\mu}_t^{2n}, \widetilde{\nu}_t^{2n}) \xrightarrow{\mathrm{P}} 0 \quad \text{as } n \to \infty. \tag{4.52}$$

Here the idea is also the same as before. First we calculate the distribution of the vector of weights

$$\widetilde{\mu}(t) = (\widetilde{\mu}_1^n(t), \cdots, \widetilde{\mu}_n^n(t)) \tag{4.53}$$

which is a generalized Dirichlet distribution (we can think of the one defined in (3.18) as the standard one) supported on the simplex

$$S(1/2) = \left\{ (y_1, \cdots, y_{n-1}) \middle| y_j \geq 0 \text{ for all } j, \sum_{j=1}^{n-1} y_j \leq 1/2 \right\}. \tag{4.54}$$

Then for each $1 \leq k \leq n$, we find

$$\sum_{j=1}^{k} \widetilde{\mu}_j^n(t) \sim \mathrm{Beta}^{(0,\frac{1}{2})}\left(\frac{k\beta}{2}, \frac{(n-k)\beta}{2}\right), \tag{4.55}$$

where $\mathrm{Beta}^{(0,\frac{1}{2})}$ denotes the generalized Beta distribution supported on $[0, \frac{1}{2}]$. Since we have

$$d(\widetilde{\mu}_t^{2n}, \widetilde{\nu}_t^{2n}) \leq \max \left|\sum_{j=1}^{k} \widetilde{\mu}_j^{(n)}(t) - \frac{k}{2n}\right|, \tag{4.56}$$

the analysis proceeds as in Section 3.2. Consequently, when we combine (4.51) and (4.52), we conclude that

$$d(\widetilde{\nu}_t^{2n}, \widetilde{\mu}_t) \xrightarrow{\mathrm{P}} 0 \quad \text{as } n \to \infty. \tag{4.57}$$

We are now ready to state the main result for the β-Laguerre process. To this end, introduce the empirical singular value process

$$\bar{\nu}_t^n = \frac{1}{n} \sum_{i=1}^{n} \delta_{\sigma_j^{(n)}(t)} \tag{4.58}$$

associated with $(L_{\beta,a}^{(n)}(t))_{t \geq 0}$. Also, let

$$d\,\bar{\mu}_t(x) = \frac{\sqrt{4\rho(t) - x^2}}{\pi\rho(t)}\, 1_{[\,0, 2\sqrt{\rho(t)}\,]}(x)\, dx. \tag{4.59}$$

We will consider $\bar{\nu}_t^n$ and $\bar{\mu}_t$ as measures on $\mathbb{R}_+ = [0, \infty)$.

Theorem 4.2.4. *For each $t > 0$, the sequence $(\bar{\nu}_t^n)_{n \geq 1}$ converges weakly, in probability, to the probability measure $\bar{\mu}_t$.*

Proof. We will deduce this result from (4.57). For this purpose, it is more convenient to use the bounded Lipschitz metric, which is equivalent to the Lévy-Prohorov metric [Dud]. In order to write down the expression for this metric, denote by $BL(S)$ the class of bounded functions $f : S \longrightarrow \mathbb{R}$ on a complete metric space S which are Lipschitz. For $f \in BL(S)$, define the Lipschitz semi-norm

$$\|f\|_{L(S)} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)} \tag{4.60}$$

and put $\|f\|_{BL(S)} = \|f\|_{L(S)} + \|f\|_{L^\infty(S)}$ where $\|f\|_{L^\infty(S)}$ is the sup-norm. Then $\|f\|_{BL(S)}$ is a norm and $(BL(S), \|\cdot\|_{BL(S)})$ is a Banach space. With these notations, the bounded Lipschitz distance between the two measure $\widetilde{\nu}_t^{2n}$ and $\widetilde{\mu}_t$ is given by

$$d_{BL(\mathbb{R})}(\widetilde{\nu}_t^{2n}, \widetilde{\mu}_t) = \sup_{f \in BL_1(\mathbb{R})} \Big| \int_{\mathbb{R}} f\, d\,\widetilde{\nu}_t^{2n} - \int_{\mathbb{R}} f\, d\,\widetilde{\mu}_t \Big| \tag{4.61}$$

where the supremum is taken over

$$BL_1(\mathbb{R}) = \{f \in BL(\mathbb{R}) \mid \|f\|_{BL(\mathbb{R})} \leq 1\}. \tag{4.62}$$

Now let $BL_1^e(\mathbb{R}) = \{f \in BL_1(\mathbb{R}) \mid f \text{ is even}\}$, then clearly

$$d_{BL(\mathbb{R})}(\widetilde{\nu}_t^{2n}, \widetilde{\mu}_t) \geq \sup_{f\in BL_1^e(\mathbb{R})}\Big|\int_{\mathbb{R}} f d\,\widetilde{\nu}_t^{2n} - \int_{\mathbb{R}} f d\,\widetilde{\mu}_t\Big|. \tag{4.63}$$

But for $f \in BL_1^e(\mathbb{R})$, it follows from the definition of the two measures that

$$\int_{\mathbb{R}} f d\,\widetilde{\nu}_t^{2n} = \frac{1}{n}\sum_{j=1}^{n} f(\sigma_j^{(n)}(t)) = \int_{\mathbb{R}_+} f d\,\bar{\nu}_t^n, \tag{4.64}$$

while

$$\int_{\mathbb{R}} f d\,\widetilde{\mu}_t = \int_{\mathbb{R}_+} f(x) d\,\bar{\mu}_t. \tag{4.65}$$

As we can identify the space $BL_1^e(\mathbb{R})$ with $BL_1(\mathbb{R}_+)$, when we combine (4.63)-(4.65), the result is

$$\begin{aligned} d_{BL(\mathbb{R})}(\widetilde{\nu}_t^{2n}, \widetilde{\mu}_t) &\geq \sup_{f\in BL_1(\mathbb{R}_+)}\Big|\int_{\mathbb{R}_+} f d\,\bar{\nu}_t^n - \int_{\mathbb{R}_+} f d\,\bar{\mu}_t\Big| \\ &= d_{BL(\mathbb{R}_+)}(\bar{\nu}_t^n, \bar{\mu}_t). \end{aligned} \tag{4.66}$$

Hence it follows from (4.57) and (4.66) that $d_{BL(\mathbb{R}_+)}(\bar{\nu}_t^n, \bar{\mu}_t) \xrightarrow{\mathrm{P}} 0 \quad \text{as } n \to \infty.$

$\square$

Remark 4.2.5. If we let

$$\bar{\mu}_t^n = 2\sum_{j=1}^{n} \widetilde{\mu}_j^n(t)\delta_{\sigma_j^{(n)}(t)} = \sum_{j=1}^{n} f_j(1,t)^2\delta_{\sigma_j^{(n)}(t)}, \tag{4.67}$$

then following the same argument as in the proof of Theorem 4.2.4, we also have

$$d_{BL(\mathbb{R}_+)}(\bar{\mu}_t^n, \bar{\mu}_t) \xrightarrow{\mathrm{P}} 0 \quad \text{as } n \to \infty. \tag{4.68}$$

The measure valued process $(\bar{\mu}_t^n)_{t\geq 0}$, however, is the spectral measure process of the square root $\left(\sqrt{J_{\beta,a}^{(n)}(t)}\right)_{t\geq 0}$ of the scaled β-Wishart process, where $J_{\beta,a}^{(n)}(t)$ is defined in (4.17).

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L.-C. Li, Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA

E-mail address: `luenli@math.psu.edu`